Tensor products and Minkowski sums of Mirković-Vilonen polytopes

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Abstract

The purpose of this paper is to prove that the Mirković-Vilonen (MV for short) polytope corresponding to the tensor product of two arbitrary MV polytopes is contained in the Minkowski sum of these two MV polytopes. This generalizes the result in our previous paper [KNS], which was obtained under the assumption that the first tensor factor is an extremal MV polytope.

1 Introduction.

In our previous paper [KNS], we proved that the Mirković-Vilonen (MV for short) polytope corresponding to the tensor product of two MV polytopes is contained in the Minkowski sum of the two MV polytopes, under the assumption that the first (i.e., left) tensor factor is an extremal MV polytope. The purpose of the present paper is to prove the same result for arbitrary two MV polytopes, without any assumption on the first tensor factor.

*Supported in part by JSPS Research Fellowships for Young Scientists (No. 20740011).
†Supported in part by Grant-in-Aid for Scientific Research (No. 20540006), JSPS.
‡Supported in part by JSPS Research Fellowships for Young Scientists (No. 19740004).

Following the notation of [KNS], let G be a complex semisimple algebraic group with Lie algebra $\mathfrak{g}$, and T a maximal torus with Lie algebra $\mathfrak{h}$. For a dominant coweight $\lambda \in X_*(T) := \mathrm{Hom}(\mathbb{C}^*, T)$ for G, let $\mathcal{MV}(\lambda)$ denote the set of all MV polytopes P of highest vertex λ such that $P \subset \mathrm{Conv}(W \cdot \lambda) \subset \mathfrak{h}_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$; the set $\mathcal{MV}(\lambda)$ gives a realization of the crystal basis $\mathcal{B}(\lambda)$ of the irreducible highest weight module of highest weight λ over the quantized universal enveloping algebra $U_q(\mathfrak{g}^\vee)$ of the (Langlands) dual Lie algebra $\mathfrak{g}^\vee$ of $\mathfrak{g}$.

Let λ_1, $\lambda_2 \in X_*(T)$ be dominant coweights, and $P_1 \in \mathcal{MV}(\lambda_1)$, $P_2 \in \mathcal{MV}(\lambda_2)$. If we consider the tensor product

$$P_1 \otimes P_2 \in \mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2) \cong \mathcal{B}(\lambda_1) \otimes \mathcal{B}(\lambda_2),$$

then there exists a unique dominant coweight $\lambda \in X_*(T)$ and embedding $\iota_\lambda : \mathcal{MV}(\lambda)$ ($\cong \mathcal{B}(\lambda)$) $\hookrightarrow \mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2)$ of crystals such that $P_1 \otimes P_2 = \iota_\lambda(P)$ for some $P \in \mathcal{MV}(\lambda)$. Now our main result (Theorem 3.1.1) states that in $\mathfrak{h}_{\mathbb{R}}$, we have the inclusion $P \subset P_1 + P_2$, where $P_1 + P_2$ is the Minkowski sum of the MV polytopes P_1 and P_2; in [KNS], we proved the same assertion under the assumption that P_1 is an extremal MV polytope.

We should mention that our method of proof for Theorem 3.1.1 is quite different from the one of [KNS], in which we used twisted products of (open dense subsets of) MV cycles in the twisted product of two affine Grassmannians. In fact, we make use of a description (Proposition 2.5.1) of MV polytopes in terms of Kashiwara data for $U_q(\mathfrak{g}^\vee)$, due to Ehrig, and reduce the problem to proving an inequality (Proposition 3.2.2) between Gelfand-Goresky-MacPherson-Serganova (GGMS for short) data, or equivalently, between Berenstein-Zelevinsky (BZ for short) data. This inequality (more precisely, Proposition A.1.2 in the Appendix) can be regarded as a generalization to an arbitrary semisimple Lie algebra of the inequality of [Kam2, Proposition 2.7] in the case of type A;[1] in [Kam2], the inequality was used to study the irreducible components of certain fibers of the convolution morphism for the affine Grassmannian.

Since the inequality above is the most important ingredient, we provide two different proofs: one proof (given in Subsection 3.2) is based on the agreement, due to Kamnitzer [Kam1], of the Lusztig-Berenstein-Zelevinsky (LBZ for short) and Braverman-Finkelberg-Gaitsgory (BFG for short) crystal structures for $U_q(\mathfrak{g}^\vee)$ on the set of MV polytopes; another proof (given in the Appendix) is a purely geometric one based on the original definition of MV polytopes by Anderson [A].

Once the inequality above is obtained, Theorem 3.1.1 follows easily from the tensor product rule for the action of lowering Kashiwara operators for $U_q(\mathfrak{g}^\vee)$.

[1]After finishing this paper, we were informed by Kamnitzer that he also had (but, never really wrote down) a proof of this inequality in the general case along the lines of our geometric proof in the Appendix

This paper is organized as follows. In Section 2, we first recall the basic notation and standard facts concerning MV polytopes. Next, we review the relation of MV polytopes with MV cycles in the affine Grassmannian, and also the LBZ (= BFG) crystal structure on the set of MV polytopes. Furthermore, we give a description of MV polytopes in terms of Kashiwara data, due to Ehrig; we include a short proof of it, which uses Kamnitzer's result. In Section 3, we first state our main result (Theorem 3.1.1). Next, we prove an inequality (Proposition 3.2.2) between GGMS (or BZ) data, which is a key to our proof of Theorem 3.1.1. Finally, by combining the above with the tensor product rule for crystals, we prove Theorem 3.1.1. In the Appendix, using the geometry of the affine Grassmannian, we give another proof of the inequality above (or, a slightly strengthened form of it).

2 Mirković-Vilonen polytopes.

2.1 Basic notation. Let G be a complex connected semisimple algebraic group, T a maximal torus, B a Borel subgroup containing T, and U the unipotent radical of B; we choose the convention that the roots in B are the negative ones. Let $X_*(T)$ denote the (integral) coweight lattice $\mathrm{Hom}(\mathbb{C}^*, T)$ for G, and $X_*(T)_+$ the set of dominant (integral) coweights for G; we regard the coweight lattice $X_*(T)$ as an additive subgroup of a real form $\mathfrak{h}_\mathbb{R} := \mathbb{R} \otimes_\mathbb{Z} X_*(T)$ of the Lie algebra $\mathfrak{h}$ of the maximal torus T. We denote by $G^\vee$ the (complex) Langlands dual group of G.

Denote by $\mathfrak{g}$ the Lie algebra of G, which is a complex semisimple Lie algebra. Let

$$\Big(A = (a_{ij})_{i,j\in I},\, \Pi := \big\{\alpha_j\big\}_{j\in I},\, \Pi^\vee := \big\{h_j\big\}_{j\in I},\, \mathfrak{h}^*,\, \mathfrak{h}\Big)$$

be the root datum of $\mathfrak{g}$, where $A = (a_{ij})_{i,j\in I}$ is the Cartan matrix, $\mathfrak{h}$ is the Cartan subalgebra, $\Pi := \big\{\alpha_j\big\}_{j\in I} \subset \mathfrak{h}^* := \mathrm{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C})$ is the set of simple roots, and $\Pi^\vee := \big\{h_j\big\}_{j\in I} \subset \mathfrak{h}$ is the set of simple coroots; note that $\langle h_i, \alpha_j\rangle = a_{ij}$ for $i, j \in I$, where $\langle\cdot, \cdot\rangle$ denotes the canonical pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$, and that $\mathfrak{h}_\mathbb{R} = \sum_{j\in I} \mathbb{R}h_j \subset \mathfrak{h}$. Also, for $h, h' \in \mathfrak{h}_\mathbb{R}$, we write $h' \ge h$ if $h' - h \in Q_+^\vee := \sum_{j\in I} \mathbb{Z}_{\ge 0} h_j$. Let $W := \langle s_j \mid j \in I\rangle$ be the Weyl group of $\mathfrak{g}$, where s_j, $j \in I$, are the simple reflections, with length function $\ell : W \to \mathbb{Z}_{\ge 0}$, the identity element $e \in W$, and the longest element $w_0 \in W$; we denote by $\le$ the (strong) Bruhat order on W. Let $\mathfrak{g}^\vee$ denote the Lie algebra of the Langlands dual group $G^\vee$ of G, which is the complex semisimple Lie algebra associated to the root datum

$$\Big({}^tA = (a_{ji})_{i,j\in I},\, \Pi^\vee = \big\{h_j\big\}_{j\in I},\, \Pi = \big\{\alpha_j\big\}_{j\in I},\, \mathfrak{h},\, \mathfrak{h}^*\Big);$$

note that the Cartan subalgebra of $\mathfrak{g}^\vee$ is $\mathfrak{h}^*$, not $\mathfrak{h}$. Let $U_q(\mathfrak{g}^\vee)$ be the quantized universal enveloping algebra of $\mathfrak{g}^\vee$ over $\mathbb{C}(q)$. For a dominant coweight $\lambda \in X_*(T)_+ \subset \mathfrak{h}_\mathbb{R}$, denote by $V(\lambda)$ the irreducible highest weight $U_q(\mathfrak{g}^\vee)$-module of highest weight λ, and by $\mathcal{B}(\lambda)$ the crystal basis of $V(\lambda)$.

2.2 Mirković-Vilonen polytopes. In this subsection, following [Kam3], we recall a (combinatorial) characterization of Mirković-Vilonen (MV for short) polytopes; the relation between this characterization and the original (geometric) definition of MV polytopes given by Anderson [A] will be explained in §2.3.

As in §2.1, we assume that $\mathfrak{g}$ is a complex semisimple Lie algebra. Let $\mu_\bullet = (\mu_w)_{w \in W}$ be a collection of elements of $X_*(T) \subset \mathfrak{h}_\mathbb{R} = \sum_{j \in I} \mathbb{R} h_j$. We call $\mu_\bullet = (\mu_w)_{w \in W}$ a Gelfand-Goresky-MacPherson-Serganova (GGMS) datum if it satisfies the condition that

$$x^{-1} \cdot \mu_z - x^{-1} \cdot \mu_x \in Q_+^\vee \qquad \text{for all } x,\, z \in W. \tag{2.2.1}$$

It follows by induction with respect to the (weak) Bruhat order on W that $\mu_\bullet = (\mu_w)_{w \in W}$ is a GGMS datum if and only if

$$\mu_{ws_i} - \mu_w \in \mathbb{Z}_{\geq 0}\,(w \cdot h_i) \quad \text{for every } w \in W \text{ and } i \in I. \tag{2.2.2}$$

Remark 2.2.1. Let $\mu_\bullet = (\mu_w)_{w \in W}$ be a GGMS datum, and take an arbitrary $w \in W$. We see from (2.2.1) (with $x = w_0$ and $z = w$) that $w_0^{-1} \cdot \mu_w - w_0^{-1} \cdot \mu_{w_0} \in Q_+^\vee$, which implies that μ_w is contained in $\mu_{w_0} - Q_+^\vee$. Since $\mu_{w_0} \in X_*(T)$, we deduce that $w^{-1} \cdot \mu_w$ is contained in $\mu_{w_0} - Q^\vee$, where we set $Q^\vee := \sum_{j \in I} \mathbb{Z} h_j$. Hence

$$z^{-1} \cdot \mu_z - x^{-1} \cdot \mu_x \in Q^\vee \quad \text{for all } x,\, z \in W.$$

Remark 2.2.2. Let $\mu_\bullet^{(1)} = (\mu_w^{(1)})_{w \in W}$ and $\mu_\bullet^{(2)} = (\mu_w^{(2)})_{w \in W}$ be GGMS data. Then, it is obvious from the definition of GGMS data (i.e., from (2.2.2)) that the (componentwise) sum

$$\mu_\bullet^{(1)} + \mu_\bullet^{(2)} := (\mu_w^{(1)} + \mu_w^{(2)})_{w \in W}$$

of $\mu_\bullet^{(1)}$ and $\mu_\bullet^{(2)}$ is also a GGMS datum.

Following [Kam3] and [Kam1], to each GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$, we associate a convex polytope $P(\mu_\bullet) \subset \mathfrak{h}_\mathbb{R}$ by:

$$P(\mu_\bullet) = \bigcap_{w \in W} \bigl\{ v \in \mathfrak{h}_\mathbb{R} \mid w^{-1} \cdot v - w^{-1} \cdot \mu_w \in \textstyle\sum_{j \in I} \mathbb{R}_{\geq 0} h_j \bigr\}; \tag{2.2.3}$$

the polytope $P(\mu_\bullet)$ is called a pseudo-Weyl polytope with GGMS datum $\mu_\bullet$. Note that the GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$ is determined uniquely by the convex polytope $P(\mu_\bullet)$. Also, we know from [Kam3, Proposition 2.2] that the set of vertices of the polytope $P(\mu_\bullet)$ is given by the collection $\mu_\bullet = (\mu_w)_{w \in W}$ (possibly, with repetitions). In particular, we have

$$P(\mu_\bullet) = \mathrm{Conv}\,\bigl\{ \mu_w \mid w \in W \bigr\}, \tag{2.2.4}$$

where for a subset X of $\mathfrak{h}_\mathbb{R}$, $\mathrm{Conv}\,X$ denotes the convex hull in $\mathfrak{h}_\mathbb{R}$ of X.

In the proof of Theorem 3.1.1 below, we need the following lemma about Minkowski sums of pseudo-Weyl polytopes.

Lemma 2.2.3 ([Kam3, Lemma 6.1]). *Let $P_1 = P(\mu_\bullet^{(1)})$ and $P_2 = P(\mu_\bullet^{(2)})$ be pseudo-Weyl polytopes with GGMS data $\mu_\bullet^{(1)} = (\mu_w^{(1)})_{w\in W}$ and $\mu_\bullet^{(2)} = (\mu_w^{(2)})_{w\in W}$, respectively. Then, the Minkowski sum*

$$P_1 + P_2 := \{v_1 + v_2 \mid v_1 \in P_1,\, v_2 \in P_2\}$$

of the pseudo-Weyl polytopes P_1 and P_2 is identical to the pseudo-Weyl polytope $P(\mu_\bullet^{(1)} + \mu_\bullet^{(2)})$ having GGMS datum $\mu_\bullet^{(1)} + \mu_\bullet^{(2)} = (\mu_w^{(1)} + \mu_w^{(2)})_{w\in W}$ (see Remark 2.2.2).

Furthermore, we need to recall from [Kam3, §2.3] the notion of Berenstein-Zelevinsky (BZ for short) data. We set $\Gamma := \{w \cdot \Lambda_j \mid w \in W,\, j \in I\}$, where Λ_j, $j \in I$, are the fundamental weights for $\mathfrak{g}$. Let $P = P(\mu_\bullet)$ be a pseudo-Weyl polytope with GGMS datum $\mu_\bullet = (\mu_w)_{w\in W}$. For each $\gamma \in \Gamma$, we set

$$M_\gamma := \langle \mu_w,\, w \cdot \Lambda_j \rangle \quad \text{if } \gamma = w \cdot \Lambda_j \text{ for some } w \in W \text{ and } j \in I;$$

note that the number M_γ does note depend on the expression $\gamma = w \cdot \Lambda_j$, $w \in W$, $j \in I$, of $\gamma \in \Gamma$. We call the collection $M_\bullet = (M_\gamma)_{\gamma\in\Gamma}$ the BZ datum of the pseudo-Weyl polytope P. We know from [Kam3, Proposition 2.2] that

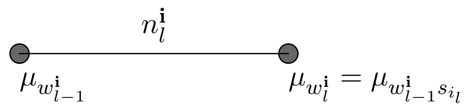
$$P = P(\mu_\bullet) = \{v \in \mathfrak{h}_\mathbb{R} \mid \langle v,\, \gamma \rangle \geq M_\gamma \text{ for all } \gamma \in \Gamma\}. \tag{2.2.5}$$

Since the proof of [NS, Lemma 4.5.4] for MV polytopes works equally well for pseudo-Weyl polytopes, we have the following.

Lemma 2.2.4. *Let P and P' be pseudo-Weyl polytopes with BZ data $M_\bullet = (M_\gamma)_{\gamma\in\Gamma}$ and $M'_\bullet = (M'_\gamma)_{\gamma\in\Gamma}$, respectively. Then, $P \subset P'$ if and only if $M_\gamma \geq M'_\gamma$ for all $\gamma \in \Gamma$.*

Let $R(w_0)$ denote the set of all reduced words for w_0, that is, all sequences $(i_1, i_2, \ldots, i_m)$ of elements of I such that $s_{i_1} s_{i_2} \cdots s_{i_m} = w_0$, where m is the length $\ell(w_0)$ of the longest element w_0. Let $\mathbf{i} = (i_1, i_2, \ldots, i_m) \in R(w_0)$ be a reduced word for w_0. We set $w_l^{\mathbf{i}} := s_{i_1} s_{i_2} \cdots s_{i_l} \in W$ for $0 \leq l \leq m$. For a GGMS datum $\mu_\bullet = (\mu_w)_{w\in W}$, define integers (called the lengths of edges) $n_l^{\mathbf{i}} = n_l^{\mathbf{i}}(\mu_\bullet) \in \mathbb{Z}_{\geq 0}$, $1 \leq l \leq m$, via the following "length formula" (see [Kam3, Eq. (8)] and (2.2.2) above):

$$\mu_{w_l^{\mathbf{i}}} - \mu_{w_{l-1}^{\mathbf{i}}} = n_l^{\mathbf{i}}\, w_{l-1}^{\mathbf{i}} \cdot h_{i_l}. \tag{2.2.6}$$



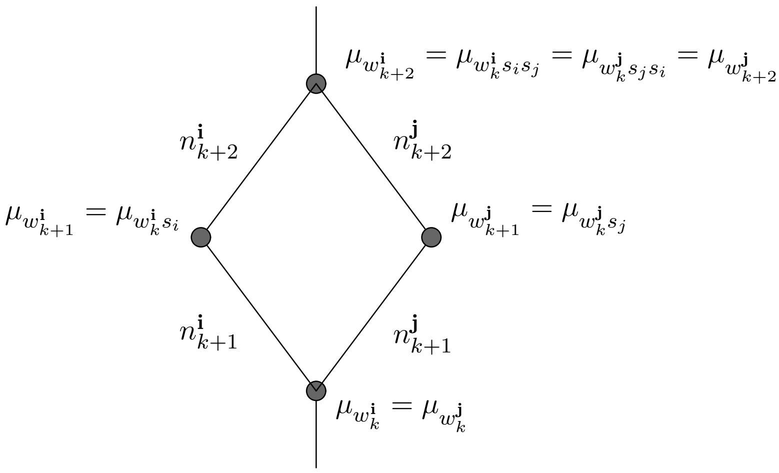
Now we are ready to give a (combinatorial) characterization of Mirković-Vilonen (MV) polytopes, due to Kamnitzer [Kam3]. This result holds for an arbitrary complex semisimple

Lie algebra $\mathfrak{g}$, but we give its precise statement only in the case that $\mathfrak{g}$ is simply-laced since we do not make use of it in this paper; we merely mention that when $\mathfrak{g}$ is not simply-laced, there are also conditions on the lengths $n_l^{\mathbf{i}}$, $1 \le l \le m$, $\mathbf{i} \in R(w_0)$, for the other possible values of a_{ij} and a_{ji} (we refer the reader to [BeZ, §3] for explicit formulas).

Definition 2.2.5. A GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$ is said to be a Mirković-Vilonen (MV) datum if it satisfies the following conditions:

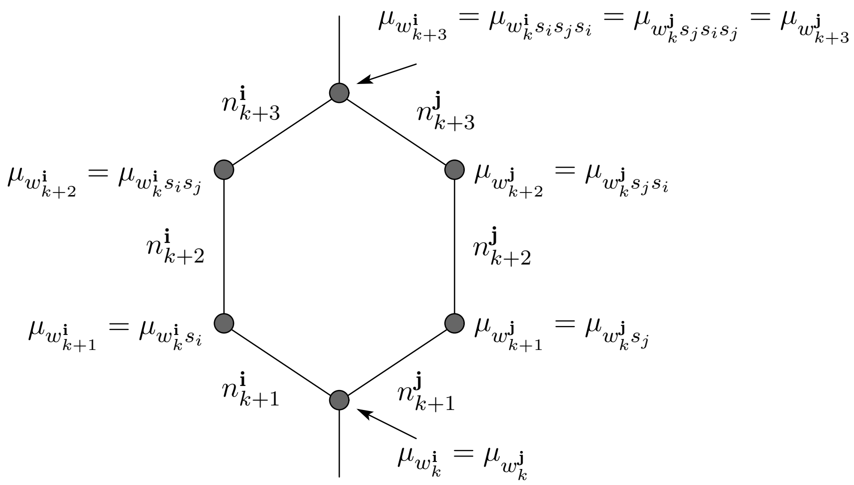
(1) If $\mathbf{i} = (i_1, i_2, \dots, i_m) \in R(w_0)$ and $\mathbf{j} = (j_1, j_2, \dots, j_m) \in R(w_0)$ are related by a 2-move, that is, if there exist indices $i, j \in I$ with $a_{ij} = a_{ji} = 0$ and an integer $0 \le k \le m-2$ such that $i_l = j_l$ for all $1 \le l \le m$ with $l \ne k+1, k+2$, and such that $i_{k+1} = j_{k+2} = i$, $i_{k+2} = j_{k+1} = j$, then there hold

$$\begin{cases} n_l^{\mathbf{i}} = n_l^{\mathbf{j}} \quad \text{for all } 1 \le l \le m \text{ with } l \ne k+1,\, k+2, \text{ and} \\ n_{k+1}^{\mathbf{i}} = n_{k+2}^{\mathbf{j}}, \quad n_{k+2}^{\mathbf{i}} = n_{k+1}^{\mathbf{j}}. \end{cases}$$



(2) If $\mathbf{i} = (i_1, i_2, \dots, i_m) \in R(w_0)$ and $\mathbf{j} = (j_1, j_2, \dots, j_m) \in R(w_0)$ are related by a 3-move, that is, if there exist indices $i, j \in I$ with $a_{ij} = a_{ji} = -1$ and an integer $0 \le k \le m-3$ such that $i_l = j_l$ for all $1 \le l \le m$ with $l \ne k+1, k+2, k+3$, and such that $i_{k+1} = i_{k+3} = j_{k+2} = i$, $i_{k+2} = j_{k+1} = j_{k+3} = j$, then there hold

$$\begin{cases} n_l^{\mathbf{i}} = n_l^{\mathbf{j}} \quad \text{for all } 1 \le l \le m \text{ with } l \ne k+1,\, k+2,\, k+3, \text{ and} \\ n_{k+1}^{\mathbf{j}} = n_{k+2}^{\mathbf{i}} + n_{k+3}^{\mathbf{i}} - \min\bigl(n_{k+1}^{\mathbf{i}}, n_{k+3}^{\mathbf{i}}\bigr), \\ n_{k+2}^{\mathbf{j}} = \min\bigl(n_{k+1}^{\mathbf{i}}, n_{k+3}^{\mathbf{i}}\bigr), \\ n_{k+3}^{\mathbf{j}} = n_{k+1}^{\mathbf{i}} + n_{k+2}^{\mathbf{i}} - \min\bigl(n_{k+1}^{\mathbf{i}}, n_{k+3}^{\mathbf{i}}\bigr). \end{cases}$$



The pseudo-Weyl polytope $P(\mu_\bullet)$ with GGMS datum $\mu_\bullet = (\mu_w)_{w\in W}$ (see (2.2.3)) is a Mirković-Vilonen (MV) polytope if and only if the GGMS datum $\mu_\bullet = (\mu_w)_{w\in W}$ is an MV datum (see the proof of [Kam3, Proposition 5.4] and the comment following [Kam3, Theorem 7.1]). Also, for a dominant coweight $\lambda \in X_*(T)_+ \subset \mathfrak{h}_\mathbb{R}$ and a coweight $\nu \in X_*(T) \subset \mathfrak{h}_\mathbb{R}$, an MV polytope $P = P(\mu_\bullet)$ with GGMS datum $\mu_\bullet = (\mu_w)_{w\in W}$ is an MV polytope of highest vertex λ and lowest vertex ν if and only if $\mu_{w_0} = \lambda$, $\mu_e = \nu$, and P is contained in the convex hull $\mathrm{Conv}(W \cdot \lambda)$ of the W-orbit $W \cdot \lambda \subset \mathfrak{h}_\mathbb{R}$ (see [A, Proposition 7]); we denote by $\mathcal{MV}(\lambda)_\nu$ the set of MV polytopes of highest vertex λ and lowest vertex ν. For each dominant coweight $\lambda \in X_*(T)_+ \subset \mathfrak{h}_\mathbb{R}$, we set

$$\mathcal{MV}(\lambda) := \bigsqcup_{\nu \in X_*(T)} \mathcal{MV}(\lambda)_\nu.$$

2.3 Relation between MV polytopes and MV cycles. In this subsection, we review the relation of MV polytopes with MV cycles in the affine Grassmannian.

Let us recall the definition of MV cycles in the affine Grassmannian, following [MV1], [MV2] (and [A]). Let G be a complex connected semisimple algebraic group with Lie algebra $\mathfrak{g}$, as in §2.1. Let $\mathcal{O} = \mathbb{C}[[t]]$ denote the ring of formal power series, and $\mathcal{K} = \mathbb{C}((t))$ the field of formal Laurent series (the fraction field of $\mathcal{O}$). The affine Grassmannian $\mathcal{G}r$ for G over $\mathbb{C}$ is defined to be the quotient $G(\mathcal{K})/G(\mathcal{O})$, equipped with the structure of a complex algebraic ind-scheme, where $G(\mathcal{K})$ denotes the set of $\mathcal{K}$-valued points of G, and $G(\mathcal{O}) \subset G(\mathcal{K})$ denotes the set of $\mathcal{O}$-valued points of G; we denote by $\pi : G(\mathcal{K}) \twoheadrightarrow \mathcal{G}r = G(\mathcal{K})/G(\mathcal{O})$ the natural quotient map, which is locally trivial in the Zariski topology. In what follows, for a subgroup $H \subset G(\mathcal{K})$ that is stable under the adjoint action of T and for an element w of the Weyl group $W \cong N_G(T)/T$ of G, we denote by ${}^wH$ the w-conjugate $\dot{w}H\dot{w}^{-1}$ of H, where $\dot{w} \in N_G(T)$ is a lift of $w \in W$.

Since each coweight $\nu \in X_*(T) = \mathrm{Hom}(\mathbb{C}^*, T)$ is a regular map from $\mathbb{C}^*$ to $T \subset G$, it gives a point $t^\nu \in G(\mathcal{K})$, which in turn, descends to a point $[t^\nu] \in \mathcal{G}r = G(\mathcal{K})/G(\mathcal{O})$.

For each $\nu \in X_*(T)$, we set

$$\mathcal{G}r^\nu := G(\mathcal{O})[t^\nu] \subset \mathcal{G}r,$$

the $G(\mathcal{O})$-orbit of $[t^\nu]$, which is a smooth quasi-projective algebraic variety over $\mathbb{C}$. Also, for each $\nu \in X_*(T)$ and $w \in W$, we set

$$S_\nu^w := {}^wU(\mathcal{K})[t^\nu] \subset \mathcal{G}r,$$

the ${}^wU(\mathcal{K})$-orbit of $[t^\nu]$, which is a (locally closed) ind-subscheme of $\mathcal{G}r$; we write simply S_ν for S_ν^e. Then, we know the following two kinds of decompositions of $\mathcal{G}r$ into orbits. First, we have

$$\mathcal{G}r = \bigsqcup_{\lambda \in X_*(T)_+} \mathcal{G}r^\lambda \qquad \text{(Cartan decomposition)},$$

with $\mathcal{G}r^{w\cdot\lambda} = \mathcal{G}r^\lambda$ for $\lambda \in X_*(T)_+$ and $w \in W$; note that (see, for example, [MV2, §2]) for each $\lambda \in X_*(T)_+$, the quasi-projective algebraic variety $\mathcal{G}r^\lambda$ is simply-connected, and of dimension $2\langle\lambda, \rho\rangle$, where ρ denotes half the sum of the positive roots $\alpha \in \Delta_+$ for G, i.e., $2\rho = \sum_{\alpha\in\Delta_+} \alpha$. Second, we have for each $w \in W$,

$$\mathcal{G}r = \bigsqcup_{\nu \in X_*(T)} S_\nu^w \qquad \text{(Iwasawa decomposition)}.$$

Moreover, the (Zariski) closure relations among these orbits are described as follows (see [MV2, §2 and §3]):

$$\overline{\mathcal{G}r^\lambda} = \bigsqcup_{\substack{\lambda' \in X_*(T)_+ \\ \lambda' \le \lambda}} \mathcal{G}r^{\lambda'} \qquad \text{for } \lambda \in X_*(T)_+; \tag{2.3.1}$$

$$\overline{S_\nu^w} = \bigsqcup_{\substack{\gamma \in X_*(T) \\ w^{-1}\cdot\gamma \ge w^{-1}\cdot\nu}} S_\gamma^w \qquad \text{for } \nu \in X_*(T) \text{ and } w \in W. \tag{2.3.2}$$

For $\lambda \in X_*(T)_+$, let $L(\lambda)$ denote the irreducible finite-dimensional representation of the Langlands dual group $G^\vee$ of G of highest weight λ, and $\Omega(\lambda) \subset X_*(T)$ the set of weights of $L(\lambda)$. We know from [MV2, Theorem 3.2 and Remark 3.3] that $\nu \in X_*(T)$ is an element of $\Omega(\lambda)$ if and only if $\mathcal{G}r^\lambda \cap S_\nu \neq \emptyset$, and, in this case, the intersection $\mathcal{G}r^\lambda \cap S_\nu$ is of pure dimension $\langle\lambda - \nu, \rho\rangle$.

Now we come to the definition of MV cycles in the affine Grassmannian.

Definition 2.3.1 ([MV2, §3]; see also [A, §5.3])**.** Let $\lambda \in X_*(T)_+$ and $\nu \in X_*(T)$ be such that $\mathcal{G}r^\lambda \cap S_\nu \neq \emptyset$, i.e., $\nu \in \Omega(\lambda)$. An MV cycle of highest weight λ and weight ν is defined to be an irreducible component of the (Zariski) closure of the intersection $\mathcal{G}r^\lambda \cap S_\nu$.

We denote by $\mathcal{Z}(\lambda)_\nu$ the set of MV cycles of highest weight $\lambda \in X_*(T)_+$ and weight $\nu \in X_*(T)$. Also, for each $\lambda \in X_*(T)_+$, we set

$$\mathcal{Z}(\lambda) := \bigsqcup_{\nu \in X_*(T)} \mathcal{Z}(\lambda)_\nu,$$

where $\mathcal{Z}(\lambda)_\nu := \emptyset$ if $\mathcal{G}r^\lambda \cap S_\nu = \emptyset$.

Motivated by the discovery of MV cycles in the affine Grassmannian, Anderson [A] proposed considering the "moment map images" of MV cycles as follows: Let $\lambda \in X_*(T)_+$. For an MV cycle $\mathbf{b} \in \mathcal{Z}(\lambda)$, we set

$$P(\mathbf{b}) := \mathrm{Conv}\bigl\{\nu \in X_*(T) \subset \mathfrak{h}_\mathbb{R} \mid [t^\nu] \in \mathbf{b}\bigr\},$$

and call $P(\mathbf{b}) \subset \mathfrak{h}_\mathbb{R}$ the moment map image of $\mathbf{b}$; note that $P(\mathbf{b})$ is indeed a convex polytope in $\mathfrak{h}_\mathbb{R}$.

The following theorem, due to Kamnitzer [Kam3], establishes an explicit relationship between MV polytopes and MV cycles.

Theorem 2.3.2. (1) *Let $\lambda \in X_*(T)_+$ and $\nu \in X_*(T)$ be such that $\mathcal{G}r^\lambda \cap S_\nu \neq \emptyset$. If $\mu_\bullet = (\mu_w)_{w \in W}$ denotes the GGMS datum of an MV polytope $P \in \mathcal{MV}(\lambda)_\nu$, that is, $P = P(\mu_\bullet) \in \mathcal{MV}(\lambda)_\nu$, then*

$$\mathbf{b}(\mu_\bullet) := \overline{\bigcap_{w \in W} S^w_{\mu_w}} \subset \overline{\mathcal{G}r^\lambda}$$

is an MV cycle that belongs to $\mathcal{Z}(\lambda)_\nu$.

(2) *Let $\lambda \in X_*(T)_+$. For an MV polytope $P = P(\mu_\bullet) \in \mathcal{MV}(\lambda)$ with GGMS datum $\mu_\bullet$, we set $\Phi_\lambda(P) := \mathbf{b}(\mu_\bullet)$. Then, the map $\Phi_\lambda : \mathcal{MV}(\lambda) \to \mathcal{Z}(\lambda)$, $P \mapsto \Phi_\lambda(P)$, is a bijection from $\mathcal{MV}(\lambda)$ onto $\mathcal{Z}(\lambda)$ such that $\Phi_\lambda(\mathcal{MV}(\lambda)_\nu) = \mathcal{Z}(\lambda)_\nu$ for all $\nu \in X_*(T)$ with $\mathcal{G}r^\lambda \cap S_\nu \neq \emptyset$. In particular, for each MV cycle $\mathbf{b} \in \mathcal{Z}(\lambda)$, there exists a unique MV datum $\mu_\bullet$ such that $\mathbf{b} = \mathbf{b}(\mu_\bullet)$, and in this case, the moment map image $P(\mathbf{b})$ of the MV cycle $\mathbf{b} = \mathbf{b}(\mu_\bullet)$ is identical to the MV polytope $P(\mu_\bullet) \in \mathcal{MV}(\lambda)$.*

Remark 2.3.3 ([Kam3, §2.2]). For $\nu \in X_*(T)$ and $w \in W$, the "moment map image" $P(\overline{S^w_\nu})$ of $\overline{S^w_\nu}$ is, by definition, the convex hull in $\mathfrak{h}_\mathbb{R}$ of the set $\bigl\{\gamma \in X_*(T) \subset \mathfrak{h}_\mathbb{R} \mid [t^\gamma] \in \overline{S^w_\nu}\bigr\} \subset \mathfrak{h}_\mathbb{R}$, which is identical to the (shifted) convex cone $\bigl\{v \in \mathfrak{h}_\mathbb{R} \mid w^{-1} \cdot v - w^{-1} \cdot \nu \in \sum_{j \in I} \mathbb{R}_{\geq 0} h_j\bigr\}$.

2.4 Lusztig-Berenstein-Zelevinsky crystal structure. We keep the notation and assumptions of §2.2. For an MV datum $\mu_\bullet = (\mu_w)_{w \in W}$ and $j \in I$, we denote by $f_j\mu_\bullet$ (resp., $e_j\mu_\bullet$ if $\mu_e \neq \mu_{s_j}$; note that $\mu_{s_j} - \mu_e \in \mathbb{Z}_{\geq 0} h_j$ by (2.2.2)) a unique MV datum $\mu'_\bullet = (\mu'_w)_{w \in W}$ such that $\mu'_e = \mu_e - h_j$ (resp., $\mu'_e = \mu_e + h_j$) and $\mu'_w = \mu_w$ for all $w \in W$ with $s_j w < w$ (see [Kam1, Theorem 3.5] and its proof); note that $\mu'_{w_0} = \mu_{w_0}$ and $\mu'_{s_j} = \mu_{s_j}$.

Let $\lambda \in X_*(T)_+ \subset \mathfrak{h}_\mathbb{R}$ be a dominant coweight. Following [Kam1, §6.2], we endow $\mathcal{MV}(\lambda)$ with the Lusztig-Berenstein-Zelevinsky (LBZ for short) crystal structure for $U_q(\mathfrak{g}^\vee)$ as follows. Let $P = P(\mu_\bullet) \in \mathcal{MV}(\lambda)$ be an MV polytope with GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$. The weight $\mathrm{wt}(P)$ of P is, by definition, equal to the vertex $\mu_e \in \lambda - Q_+^\vee$. For each $j \in I$, we define the lowering Kashiwara operator $f_j : \mathcal{MV}(\lambda) \cup \{\mathbf{0}\} \to \mathcal{MV}(\lambda) \cup \{\mathbf{0}\}$ and the raising Kashiwara operator $e_j : \mathcal{MV}(\lambda) \cup \{\mathbf{0}\} \to \mathcal{MV}(\lambda) \cup \{\mathbf{0}\}$ by:

$$e_j\mathbf{0} = f_j\mathbf{0} := \mathbf{0},$$

$$f_jP = f_jP(\mu_\bullet) := \begin{cases} P(f_j\mu_\bullet) & \text{if } P(f_j\mu_\bullet) \subset \mathrm{Conv}(W \cdot \lambda), \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$e_jP = e_jP(\mu_\bullet) := \begin{cases} P(e_j\mu_\bullet) & \text{if } \mu_e \neq \mu_{s_j} \text{ (i.e., } \mu_{s_j} - \mu_e \in \mathbb{Z}_{>0}h_j), \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where $\mathbf{0}$ is an additional element, not contained in $\mathcal{MV}(\lambda)$. For $j \in I$, we set $\varepsilon_j(P) := \max\{k \in \mathbb{Z}_{\geq 0} \mid e_j^k P \neq \mathbf{0}\}$ and $\varphi_j(P) := \max\{k \in \mathbb{Z}_{\geq 0} \mid f_j^k P \neq \mathbf{0}\}$.

Theorem 2.4.1 ([Kam1, Theorem 6.4])**.** *The set $\mathcal{MV}(\lambda)$, equipped with the maps* wt*, e_j, f_j $(j \in I)$, and ε_j, φ_j $(j \in I)$ above, is a crystal for $U_q(\mathfrak{g}^\vee)$. Moreover, there exists a unique isomorphism $\Psi_\lambda : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathcal{MV}(\lambda)$ of crystals for $U_q(\mathfrak{g}^\vee)$.*

Remark 2.4.2. Kamnitzer [Kam1, Theorem 4.7] proved that for each $\lambda \in X_*(T)_+$, the bijection $\Phi_\lambda : \mathcal{MV}(\lambda) \to \mathcal{Z}(\lambda)$ in Theorem 2.3.2 (2) intertwines the LBZ crystal structure on $\mathcal{MV}(\lambda)$ and the crystal structure on $\mathcal{Z}(\lambda)$ defined in [BrG] (and [BrFG]).

For $P \in \mathcal{MV}(\lambda)$ and $j \in I$, we set

$$f_j^{\max}P := f_j^{\varphi_j(P)}P.$$

2.5 Description of MV polytopes in terms of Kashiwara data. The following description of MV polytopes, due to Ehrig, is obtained as a corollary of his main result [E1, Theorem 1.1].

Proposition 2.5.1. *Let $\lambda \in X_*(T)_+ \subset \mathfrak{h}_\mathbb{R}$, and $P = P(\mu_\bullet) \in \mathcal{MV}(\lambda)$ an MV polytope with GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$. Let $x \in W$, and $x = s_{i_1}s_{i_2}\cdots s_{i_k}$ an arbitrary reduced expression of x. Then, we have*

$$\mu_x = x \cdot \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P). \tag{2.5.1}$$

Because this result itself follows easily from Kamnitzer's result [Kam1, Theorem 6.6] describing Kashiwara data in terms of BZ data, we include its short proof for the convenience of the reader; in fact, since the reverse implication is shown in the proof of [E2, Corollary 7.5], these two results are indeed equivalent.

Proof of Proposition 2.5.1. Take $i_{k+1}, i_{k+2}, \dots, i_m \in I$ in such a way that

$$\mathbf{i} = (i_1, i_2, \dots, i_k, i_{k+1}, i_{k+2}, \dots, i_m)$$

is a reduced word for the longest element $w_0 \in W$, i.e., $\mathbf{i} \in R(w_0)$. We define a sequence $(p_1, p_2, \dots, p_m) \in \mathbb{Z}_{\geq 0}^m$ of nonnegative integers (called the $\mathbf{i}$-Kashiwara datum of P) by:

$$p_l := \varphi_{i_l}(f_{i_{l-1}}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P) \quad \text{for } 1 \leq l \leq m. \tag{2.5.2}$$

Then we know from [Kam1, Theorem 6.6] that

$$p_l = M_{w_{l-1}^{\mathbf{i}} \cdot \Lambda_{i_l}} - M_{w_l^{\mathbf{i}} \cdot \Lambda_{i_l}} \quad \text{for } 1 \leq l \leq m, \tag{2.5.3}$$

where $M_\bullet = (M_\gamma)_{\gamma \in \Gamma}$ denotes the BZ datum of $P = P(\mu_\bullet) \in \mathcal{MV}(\lambda)$.

Because the $x \cdot \Lambda_j$, $j \in I$, form a basis of $\mathfrak{h}^*$, in order to show the equation (2.5.1), it suffices to prove that

$$\left\langle x \cdot \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P),\, x \cdot \Lambda_j \right\rangle = \langle \mu_x,\, x \cdot \Lambda_j \rangle \quad \text{for all } j \in I. \tag{2.5.4}$$

Fix $j \in I$. By definition, the right-hand side of (2.5.4) is equal to $M_{x \cdot \Lambda_j}$. Also, we see from the definition (2.5.2) of the $\mathbf{i}$-Kashiwara datum and the equation (2.5.3) that

$$\begin{aligned}
&\left\langle x \cdot \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P),\, x \cdot \Lambda_j \right\rangle = \left\langle \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P),\, \Lambda_j \right\rangle \\
&\quad = \left\langle \mathrm{wt}(f_{i_k}^{p_k} \cdots f_{i_2}^{p_2} f_{i_1}^{p_1} P),\, \Lambda_j \right\rangle \qquad \text{by (2.5.2)} \\
&\quad = \left\langle \mathrm{wt}\, P - \sum_{l=1}^{k} p_l h_{i_l},\, \Lambda_j \right\rangle \\
&\quad = \left\langle \mathrm{wt}\, P - \sum_{l=1}^{k} (M_{w_{l-1}^{\mathbf{i}} \cdot \Lambda_{i_l}} - M_{w_l^{\mathbf{i}} \cdot \Lambda_{i_l}}) h_{i_l},\, \Lambda_j \right\rangle \quad \text{by (2.5.3)} \\
&\quad = \langle \mathrm{wt}\, P,\, \Lambda_j \rangle - \sum_{l=1}^{k} (M_{w_{l-1}^{\mathbf{i}} \cdot \Lambda_{i_l}} - M_{w_l^{\mathbf{i}} \cdot \Lambda_{i_l}}) \delta_{i_l, j}.
\end{aligned}$$

Since $\langle \mathrm{wt}\, P,\, \Lambda_j \rangle = \langle \mu_e,\, \Lambda_j \rangle = M_{\Lambda_j}$ by the definition of $\mathrm{wt}\, P$, the left-hand side of (2.5.4) is equal to

$$M_{\Lambda_j} - \sum_{l=1}^{k} (M_{w_{l-1}^{\mathbf{i}} \cdot \Lambda_{i_l}} - M_{w_l^{\mathbf{i}} \cdot \Lambda_{i_l}}) \delta_{i_l, j}.$$

If we write the set $\{1 \le l \le k \mid i_l = j\}$ as: $\{a_1 < a_2 < \cdots < a_s\}$, then we have

$$M_{\Lambda_j} - \sum_{l=1}^{k} (M_{w^{\mathbf{i}}_{l-1} \cdot \Lambda_{i_l}} - M_{w^{\mathbf{i}}_{l} \cdot \Lambda_{i_l}}) \delta_{i_l, j} = M_{\Lambda_j} - \sum_{t=1}^{s} (M_{w^{\mathbf{i}}_{a_t - 1} \cdot \Lambda_j} - M_{w^{\mathbf{i}}_{a_t} \cdot \Lambda_j}).$$

Here, observe that

i) $M_{w^{\mathbf{i}}_{a_1-1} \cdot \Lambda_j} = M_{\Lambda_j}$ since $i_l \ne j$ for $1 \le l \le a_1 - 1$;

ii) for each $1 \le t \le s-1$, $M_{w^{\mathbf{i}}_{a_{t+1}-1} \cdot \Lambda_j} = M_{w^{\mathbf{i}}_{a_t} \cdot \Lambda_j}$ since $i_l \ne j$ for $a_t + 1 \le l \le a_{t+1} - 1$;

iii) $M_{w^{\mathbf{i}}_{a_s} \cdot \Lambda_j} = M_{w^{\mathbf{i}}_{k} \cdot \Lambda_j}$ since $i_l \ne j$ for $a_s + 1 \le l \le k$.

Therefore, we see that the left-hand side of (2.5.4) is equal to

$$\begin{aligned}
&M_{\Lambda_j} - \sum_{t=1}^{s} (M_{w^{\mathbf{i}}_{a_t-1} \cdot \Lambda_j} - M_{w^{\mathbf{i}}_{a_t} \cdot \Lambda_j}) \\
&= M_{\Lambda_j} - \left\{ (M_{w^{\mathbf{i}}_{a_1-1} \cdot \Lambda_j} - M_{w^{\mathbf{i}}_{a_1} \cdot \Lambda_j}) + \sum_{t=2}^{s-1} (M_{w^{\mathbf{i}}_{a_t-1} \cdot \Lambda_j} - M_{w^{\mathbf{i}}_{a_t} \cdot \Lambda_j}) + (M_{w^{\mathbf{i}}_{a_s-1} \cdot \Lambda_j} - M_{w^{\mathbf{i}}_{a_s} \cdot \Lambda_j}) \right\} \\
&= M_{\Lambda_j} - \left\{ (M_{\Lambda_j} - M_{w^{\mathbf{i}}_{a_1} \cdot \Lambda_j}) + \sum_{t=2}^{s-1} (M_{w^{\mathbf{i}}_{a_{t-1}} \cdot \Lambda_j} - M_{w^{\mathbf{i}}_{a_t} \cdot \Lambda_j}) + (M_{w^{\mathbf{i}}_{a_{s-1}} \cdot \Lambda_j} - M_{w^{\mathbf{i}}_{k} \cdot \Lambda_j}) \right\} \\
&= M_{w^{\mathbf{i}}_{k} \cdot \Lambda_j} = M_{x \cdot \Lambda_j} \quad \text{since } w^{\mathbf{i}}_{k} = s_{i_1} s_{i_2} \cdots s_{i_k} = x,
\end{aligned}$$

as desired. This proves the proposition. □

3 Tensor products and Minkowski sums of MV polytopes.

3.1 Main result. Let $\lambda_1, \lambda_2 \in X_*(T)_+ \subset \mathfrak{h}_{\mathbb{R}}$ be dominant coweights. Since $\mathcal{MV}(\lambda) \cong \mathcal{B}(\lambda)$ as crystals for every dominant coweight $\lambda \in X_*(T)_+ \subset \mathfrak{h}_{\mathbb{R}}$, the tensor product $\mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2)$ of the crystals $\mathcal{MV}(\lambda_1)$ and $\mathcal{MV}(\lambda_2)$ decomposes into a disjoint union of connected components as follows:

$$\mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2) \cong \bigoplus_{\lambda \in X_*(T)_+} \mathcal{MV}(\lambda)^{\oplus m^{\lambda}_{\lambda_1, \lambda_2}},$$

where $m^{\lambda}_{\lambda_1,\lambda_2} \in \mathbb{Z}_{\ge 0}$ denotes the multiplicity of $\mathcal{MV}(\lambda)$ in $\mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2)$. For each dominant coweight $\lambda \in X_*(T)_+ \subset \mathfrak{h}_{\mathbb{R}}$ such that $m^{\lambda}_{\lambda_1,\lambda_2} \ge 1$, we take (and fix) an arbitrary embedding $\iota_\lambda : \mathcal{MV}(\lambda) \hookrightarrow \mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2)$ of crystals that maps $\mathcal{MV}(\lambda)$ onto a connected component of $\mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2)$, which is isomorphic to $\mathcal{MV}(\lambda)$ as a crystal.

The following theorem is the main result of this paper; in our previous paper [KNS], we proved the same assertion under the assumption that $P_1 \in \mathcal{MV}(\lambda_1)$ is an extremal MV polytope.

Theorem 3.1.1. *Keep the notation above. Let $P_1 \in \mathcal{MV}(\lambda_1)$, $P_2 \in \mathcal{MV}(\lambda_2)$, and $P \in \mathcal{MV}(\lambda)$ be such that $\iota_\lambda(P) = P_1 \otimes P_2$ for some dominant coweight $\lambda \in X_*(T)_+ \subset \mathfrak{h}_\mathbb{R}$. Then, P is contained in the Minkowski sum $P_1 + P_2$ of the MV polytopes $P_1 \in \mathcal{MV}(\lambda_1)$ and $P_2 \in \mathcal{MV}(\lambda_2)$. Namely, we have the inclusion*

$$P \subset P_1 + P_2.$$

3.2 A key inequality and its application. Let $\lambda \in X_*(T)_+ \subset \mathfrak{h}_\mathbb{R}$ be a fixed (but arbitrary) dominant coweight, and $P = P(\mu_\bullet) \in \mathcal{MV}(\lambda)$ an (arbitrary) MV polytope with GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$.

In view of the agreement of the LBZ and BFG crystal structures on the set of MV polytopes (see Remark 2.4.2), we deduce the following fact from [BaG, Proposition 4.2]; note the convention in [BaG] that the roots in B are the positive ones, which is opposite to ours.

Fact 3.2.1. *Keep the setting above. Let $j \in I$, and assume that $f_j P \neq \mathbf{0}$. Then we have $f_j P \supset P$.*

By combining this fact with Proposition 2.5.1, we can prove the following inequality, which plays a key role in the proof of Theorem 3.1.1; in the Appendix, we will give a purely geometric proof of this inequality.

Proposition 3.2.2. *With the notation as above, let x, $z \in W$ be such that $z \leq x$ in the Bruhat order on W. Then, we have*

$$z^{-1} \cdot \mu_z \geq x^{-1} \cdot \mu_x. \tag{3.2.1}$$

Remark 3.2.3. Keep the notation and assumptions in Proposition 3.2.2. It follows from Remark 2.2.1 and the definition of the order $\geq$ on $\mathfrak{h}_\mathbb{R}$ that

$$\begin{aligned} z^{-1} \cdot \mu_z \geq x^{-1} \cdot \mu_x \quad &\Leftrightarrow \quad \langle z^{-1} \cdot \mu_z, \Lambda_j \rangle \geq \langle x^{-1} \cdot \mu_x, \Lambda_j \rangle \quad \text{for all } j \in I. \\ &\Leftrightarrow \quad M_{z \cdot \Lambda_j} \geq M_{x \cdot \Lambda_j} \quad \text{for all } j \in I. \end{aligned}$$

Remark 3.2.4. It is well-known (see, for example, [BjB, Theorem 2.6.1]) that for x, $z \in W$, $z \leq x$ in the Bruhat order on W if and only if $zW_{\Lambda_j} \leq xW_{\Lambda_j}$ in the Bruhat order on the cosets W/W_{Λ_j} modulo the stabilizer W_{Λ_j} of Λ_j in W for all $j \in I$. Moreover, if we are in the case of type A, then it is well-known (see, for example, [FZ, §3.2]) that for each (fixed) $j \in I$, $z \cdot \Lambda_j - x \cdot \Lambda_j \in Q_+ := \sum_{j \in I} \mathbb{Z}_{\geq 0} \alpha_j$ if and only if $zW_{\Lambda_j} \leq xW_{\Lambda_j}$ in the Bruhat order on W/W_{Λ_j}. Thus, the inequality (3.2.1) seems closely related to [Kam2, Proposition 2.7] in the case of type A. In fact, in the Appendix, we prove a (slightly) strengthened form (Proposition A.1.2), which can be regarded as a generalization of [Kam2, Proposition 2.7].

Proof of Proposition 3.2.2. For $z, x \in W$ such that $z \le x$, there exists a sequence $z = x_0 < x_1 < \cdots < x_s = x$ of elements in W such that $\ell(x_t) = \ell(x_{t-1}) + 1$ for $1 \le t \le s$ by the chain property (see, for example, [BjB, Theorem 2.2.6]). Hence we may assume that $\ell(x) = \ell(z)+1$.

Let $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of x. Because $z < x$ and $\ell(x) = \ell(z) + 1$, it follows from the strong exchange property (see, for example, [BjB, Theorem 1.4.3]) that z has a reduced expression of the form: $z = s_{i_1} \cdots s_{i_{l-1}} s_{i_{l+1}} \cdots s_{i_k}$ for some $1 \le l \le k$.

Case 1. Suppose that $l = 1$; in this case, we have $x = s_{i_1} z > z = s_{i_2} \cdots s_{i_k}$. Then, we see from Proposition 2.5.1 that

$$x^{-1} \cdot \mu_x = \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P), \qquad z^{-1} \cdot \mu_z = \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} P).$$

Now we set $P' = P(\mu'_\bullet) := f_{i_1}^{\max} P \in \mathcal{MV}(\lambda)$, where $\mu'_\bullet = (\mu'_w)_{w \in W}$ denotes the GGMS datum of P'. Then, again from Proposition 2.5.1, we see that

$$x^{-1} \cdot \mu_x = \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} \underbrace{f_{i_1}^{\max} P}_{=P'}) = z^{-1} \cdot \mu'_z \tag{3.2.2}$$

since $z = s_{i_2} \cdots s_{i_k}$. Because $P' = f_{i_1}^{\max} P \supset P$ by Fact 3.2.1, it follows immediately from (2.2.3) that $z^{-1} \cdot \mu_z - z^{-1} \cdot \mu'_z \in \sum_{j \in I} \mathbb{R}_{\ge 0} h_j$. Combining this and (3.2.2), we obtain

$$z^{-1} \cdot \mu_z - x^{-1} \cdot \mu_x \in \sum_{j \in I} \mathbb{R}_{\ge 0} h_j.$$

Since $z^{-1} \cdot \mu_z - x^{-1} \cdot \mu_x \in Q^\vee$ by Remark 2.2.1, we conclude that $z^{-1} \cdot \mu_z - x^{-1} \cdot \mu_x \in Q_+^\vee$, which implies that $z^{-1} \cdot \mu_z \ge x^{-1} \cdot \mu_x$.

Case 2. Suppose that $l \ge 2$. We set $P'' = P(\mu''_\bullet) := f_{i_{l-1}}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P \in \mathcal{MV}(\lambda)$, where $\mu''_\bullet = (\mu''_w)_{w \in W}$ denotes the GGMS datum of P'', and set $x'' := s_{i_l} s_{i_{l+1}} \cdots s_{i_k}$, $z'' := s_{i_{l+1}} \cdots s_{i_k}$. Then we see from Proposition 2.5.1 that

$$\begin{aligned} x^{-1} \cdot \mu_x &= \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_{l+1}}^{\max} f_{i_l}^{\max} \underbrace{f_{i_{l-1}}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P}_{=P''}) \\ &= \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_{l+1}}^{\max} f_{i_l}^{\max} P'') = (x'')^{-1} \cdot \mu''_{x''}, \end{aligned}$$

$$\begin{aligned} z^{-1} \cdot \mu_z &= \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_{l+1}}^{\max} \underbrace{f_{i_{l-1}}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P}_{=P''}) \\ &= \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_{l+1}}^{\max} P'') = (z'')^{-1} \cdot \mu''_{z''}. \end{aligned}$$

Consequently, by applying the result in Case 1 (with $x = x''$ and $z = z''$), we obtain $(z'')^{-1} \cdot \mu''_{z''} \geq (x'')^{-1} \cdot \mu''_{x''}$, and hence

$$z^{-1} \cdot \mu_z = (z'')^{-1} \cdot \mu''_{z''} \geq (x'')^{-1} \cdot \mu''_{x''} = x^{-1} \cdot \mu_x,$$

as desired. This proves the proposition. $\square$

Proposition 3.2.5. *Keep the setting above. Let $x \in W$, and let $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ be an arbitrary reduced expression of x. Suppose that $f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P \neq \mathbf{0}$ for some $c_1, c_2, \ldots, c_k \in \mathbb{Z}_{\geq 0}$. Then,*

$$\mathrm{wt}(f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P) \geq \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P). \tag{3.2.3}$$

Proof. We show the assertion by induction on $\ell(x)$. If $\ell(x) = 0$ or 1, then the assertion is obvious. Hence we assume that $\ell(x) \geq 2$.

Case 1. Suppose that $c_1 = 0$; in this case, we have $f_{i_k}^{c_\ell} \cdots f_{i_2}^{c_2} P = f_{i_k}^{c_\ell} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P \neq \mathbf{0}$. By the induction hypothesis, we have

$$\mathrm{wt}(f_{i_k}^{c_\ell} \cdots f_{i_2}^{c_2} P) \geq \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} P). \tag{3.2.4}$$

It follows from Proposition 3.2.2 (with $z = s_{i_1} x = s_{i_2} \cdots s_{i_k}$) that $(s_{i_1} x)^{-1} \cdot \mu_{s_{i_1} x} \geq x^{-1} \cdot \mu_x$. Also, by Proposition 2.5.1,

$$(s_{i_1} x)^{-1} \cdot \mu_{s_{i_1} x} = \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} P), \qquad x^{-1} \cdot \mu_x = \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P).$$

Therefore, we obtain

$$\mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} P) \geq \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P). \tag{3.2.5}$$

Combining (3.2.4) and (3.2.5), we conclude that

$$\mathrm{wt}(f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P) = \mathrm{wt}(f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} P) \geq \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P),$$

as desired.

Case 2. Suppose that $c_1 \neq 0$. We set $P' := f_{i_1}^{c_1} P$; note that $f_{i_1}^{\max} P' = f_{i_1}^{\max} P$ by the definition of $f_{i_1}^{\max}$, and that $f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} P' = f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P \neq \mathbf{0}$ by our assumption. Then we obtain

$$\begin{aligned} \mathrm{wt}(f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P) &= \mathrm{wt}(f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} P') \\ &\geq \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P') \quad \text{by the result in Case 1} \\ &= \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P), \end{aligned}$$

as desired. This proves the proposition. $\square$

3.3 Proof of the main result. We are now in a position to give a proof of our main result.

Proof of Theorem 3.1.1. We write the GGMS data of $P_1 \in \mathcal{MV}(\lambda_1)$, $P_2 \in \mathcal{MV}(\lambda_2)$, and $P \in \mathcal{MV}(\lambda)$, respectively, as:

$$\mu_\bullet^{(1)} = (\mu_w^{(2)})_{w \in W}, \quad \mu_\bullet^{(2)} = (\mu_w^{(2)})_{w \in W}, \quad \text{and} \quad \mu_\bullet = (\mu_w)_{w \in W}.$$

We know from Lemma 2.2.3 that the Minkowski sum $P_1 + P_2$ is the pseudo-Weyl polytope $P(\mu_\bullet^{(1)} + \mu_\bullet^{(2)})$ with GGMS datum $\mu_\bullet^{(1)} + \mu_\bullet^{(2)} = (\mu_w^{(1)} + \mu_w^{(2)})_{w \in W}$. Therefore, by Lemma 2.2.4, we have

$$\begin{aligned} &P \subset P_1 + P_2 \quad \Leftrightarrow \\ &(M_{w \cdot \Lambda_j} =)\ \langle \mu_w,\, w \cdot \Lambda_j \rangle \geq \langle \mu_w^{(1)} + \mu_w^{(2)},\, w \cdot \Lambda_j \rangle\ (= M'_{w \cdot \Lambda_j}) \quad \text{for all } w \in W \text{ and } j \in I, \end{aligned}$$

where $M_\bullet = (M_\gamma)_{\gamma \in \Gamma}$ and $M'_\bullet = (M'_\gamma)_{\gamma \in \Gamma}$ are the BZ data of $P = P(\mu_\bullet)$ and $P_1 + P_2 = P(\mu_\bullet^{(1)} + \mu_\bullet^{(2)})$, respectively. Hence, in order to prove the inclusion $P \subset P_1 + P_2$, it suffices to show that

$$w^{-1} \cdot \mu_w - w^{-1} \cdot (\mu_w^{(1)} + \mu_w^{(2)}) \in \sum_{j \in I} \mathbb{R}_{\geq 0} h_j \quad \text{for all } w \in W. \tag{3.3.1}$$

Take (and fix) $w \in W$ arbitrarily, and let $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of w. Then we see from Proposition 2.5.1 that $w^{-1} \cdot \mu_w = \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P)$. Because $\iota_\lambda(P) = P_1 \otimes P_2$ by our assumption, we infer that

$$\mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P) = \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} (P_1 \otimes P_2)).$$

Here, by repeated application of the tensor product rule for the action of lowering Kashiwara operators f_j for $j \in I$ (see [Kas, Chapitre 9, Exercice 9.1]), we deduce that

$$f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} (P_1 \otimes P_2) = (f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P_1) \otimes (f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P_2)$$

for some $c_1, c_2, \ldots, c_k \in \mathbb{Z}_{\geq 0}$; note that $f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P_1 \neq \mathbf{0}$ since the left-hand side of the equation above is not equal to $\mathbf{0}$ by the definition of $f_j^{\max}$, $j \in I$. Hence we have

$$\begin{aligned} w^{-1} \cdot \mu_w &= \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} (P_1 \otimes P_2)) \\ &= \mathrm{wt}(f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P_1) + \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P_2) \\ &= \mathrm{wt}(f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P_1) + w^{-1} \cdot \mu_w^{(2)} \quad \text{by Proposition 2.5.1.} \end{aligned}$$

Also, we see again by Proposition 2.5.1 that

$$w^{-1} \cdot (\mu_w^{(1)} + \mu_w^{(2)}) = w^{-1} \cdot \mu_w^{(1)} + w^{-1} \cdot \mu_w^{(2)} = \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P_1) + w^{-1} \cdot \mu_w^{(2)}.$$

From the above, we conclude that

$$w^{-1} \cdot \mu_w - w^{-1} \cdot (\mu_w^{(1)} + \mu_w^{(2)}) = \mathrm{wt}(f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P_1) - \mathrm{wt}(f_{i_k}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} P_1);$$

the latter element is contained in $Q_+^\vee = \sum_{j \in I} \mathbb{Z}_{\geq 0} h_j$ by Proposition 3.2.5 (with $\lambda = \lambda_1$ and $P = P_1$), and hence in $\sum_{j \in I} \mathbb{R}_{\geq 0} h_j$. Thus we have proved (3.3.1), thereby completing the proof of the theorem. $\square$

Remark 3.3.1. In the course of the proof above, we showed that for each $w \in W$, $w^{-1} \cdot \mu_w - w^{-1} \cdot \mu_w^{(2)}$ is the weight of the element $f_{i_k}^{c_k} \cdots f_{i_2}^{c_2} f_{i_1}^{c_1} P_1 \in \mathcal{MV}(\lambda_1)$. Because the Weyl group W acts on the crystal $\mathcal{MV}(\lambda_1) \cong \mathcal{B}(\lambda_1)$ in a canonical way, we see that $\mu_w - \mu_w^{(2)}$ is also the weight of some element in $\mathcal{MV}(\lambda_1)$. Therefore, in the notation of §2.3, we obain $\mu_w - \mu_w^{(2)} \in \Omega(\lambda_1)$; here, we recall the (well-known) fact that $\Omega(\lambda_1) = \mathrm{Conv}(W \cdot \lambda_1) \cap (\lambda_1 + Q^\vee)$.

A Appendix: A geometric proof of the inequality (3.2.1).

The aim of the appendix is to give a purely geometric proof of the inequality (3.2.1) in Proposition 3.2.2; in fact, we prove a (slightly) strengthened form of this inequality, which can be regarded as a generalization of [Kam2, Proposition 2.7] in the case of type A to an arbitrary semisimple Lie algebra (see Remark 3.2.4). Below we use the setting of §2.3. Let $\lambda \in X_*(T)_+ \subset \mathfrak{h}_\mathbb{R}$ be a dominant coweight, and $P = P(\mu_\bullet) \in \mathcal{MV}(\lambda)$ an (arbitrary) MV polytope with GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$.

Proposition A.1.2 (cf. Proposition 3.2.2). *Fix $j \in I$ arbitrarily. Let $x, z \in W$ be such that $zW_{\Lambda_j} \leq xW_{\Lambda_j}$ in the Bruhat order on the cosets W/W_{Λ_j} modulo the stabilizer W_{Λ_j} of Λ_j in W. Then, we have $\langle \mu_z, z \cdot \Lambda_j \rangle \geq \langle \mu_x, x \cdot \Lambda_j \rangle$, that is, $M_{z \cdot \Lambda_j} \geq M_{x \cdot \Lambda_j}$.*

Proof. Note that by the definition of the Bruhat order on W/W_{Λ_j}, we may assume that $x = s_\alpha z$ for a positive root $\alpha \in \Delta_+$ such that $\langle \alpha^\vee, z \cdot \Lambda_j \rangle \geq 0$, where $\alpha^\vee \in \mathfrak{h}$ denotes the coroot corresponding to α. We divide the proof into two parts according as $\langle \mu_z, \alpha \rangle < 0$ or $\langle \mu_z, \alpha \rangle \geq 0$.

Case 1. First we assume that $\langle \mu_z, \alpha \rangle < 0$. Let L_α be the connected subgroup of G generated by T and the one-parameter unipotent subgroups corresponding to $\pm\alpha$, and L the derived (connected) subgroup of L_α, which is isomorphic to either $SL_2(\mathbb{C})$ or $PGL_2(\mathbb{C})$; note that $L[t^{s_\alpha \cdot \mu_z}] = L\dot{s}_\alpha[t^{\mu_z}] = L[t^{\mu_z}]$ since $\dot{s}_\alpha \in L$. Denote by $\mathbf{b} = \mathbf{b}(\mu_\bullet) \in \mathcal{Z}(\lambda)$ the MV cycle corresponding to the MV polytope $P = P(\mu_\bullet) \in \mathcal{MV}(\lambda)$ under the bijection $\Phi_\lambda : \mathcal{MV}(\lambda) \to \mathcal{Z}(\lambda)$ in Theorem 2.3.2. Then it follows that $[t^{\mu_z}] \in \mathbf{b}$. Since $\mathbf{b} \in \mathcal{Z}(\lambda)$

is an irreducible component of $\overline{G(\mathcal{O})[t^{\lambda}] \cap U(\mathcal{K})[t^{\mu_e}]} \subset \mathcal{G}r$, it is stable under the action of $U(\mathcal{O}) = G(\mathcal{O}) \cap U(\mathcal{K})$ and of $(L \cap U)(\mathcal{O}) \subset U(\mathcal{O})$. In particular, we have

$$(L \cap U)(\mathcal{O})[t^{\mu_z}] \subset U(\mathcal{O})[t^{\mu_z}] \subset \mathbf{b},$$

and hence

$$\overline{(L \cap U)(\mathcal{O})[t^{\mu_z}]} \subset \overline{U(\mathcal{O})[t^{\mu_z}]} \subset \overline{\mathbf{b}} = \mathbf{b}.$$

Now we observe that the stabilizer $\mathrm{Stab}_L[t^{\mu_z}]$ of $[t^{\mu_z}] \in \mathcal{G}r$ in L is equal to the intersection $L \cap \mathrm{Ad}(t^{\mu_z})G(\mathcal{O}) = L \cap t^{\mu_z} G(\mathcal{O})(t^{\mu_z})^{-1}$ in $G(\mathcal{K})$. Here, from the assumption that $\langle \mu_z, \alpha \rangle < 0$, it is easily shown (by using the connectedness of the Borel subgroup $L \cap {}^{s_\alpha}B$ of L) that $L \cap \mathrm{Ad}(t^{\mu_z})G(\mathcal{O}) = L \cap {}^{s_\alpha}B \subset L$. Moreover, thanks to the Bruhat decomposition:

$$L = \big((L \cap U)\dot{s}_\alpha(L \cap B)\big) \sqcup (L \cap B),$$

we deduce that

$$(L \cap U)(L \cap {}^{s_\alpha}B) = (L \cap U)\dot{s}_\alpha(L \cap B)\dot{s}_\alpha \subset L\dot{s}_\alpha = L$$

is an open dense subset, and hence

$$(L \cap U)[t^{\mu_z}] = (L \cap U)(L \cap {}^{s_\alpha}B)[t^{\mu_z}] \subset L[t^{\mu_z}]$$

is also an open dense subset (recall that $L \cap {}^{s_\alpha}B = \mathrm{Stab}_L[t^{\mu_z}]$). Therefore, we see that

$$[t^{s_\alpha \cdot \mu_z}] \in L[t^{s_\alpha \cdot \mu_z}] = L[t^{\mu_z}] = \overline{(L \cap U)[t^{\mu_z}]} \subset \overline{(L \cap U)(\mathcal{O})[t^{\mu_z}]} \subset \overline{U(\mathcal{O})[t^{\mu_z}]} \subset \mathbf{b}.$$

Also, by part (1) of Theorem 2.3.2, we infer that $\mathbf{b} \subset \overline{S^{s_\alpha z}_{\mu_{s_\alpha z}}} = \overline{{}^{s_\alpha z}U(\mathcal{K})[t^{\mu_{s_\alpha z}}]}$. Consequently, we obtain $[t^{s_\alpha \cdot \mu_z}] \in \overline{S^{s_\alpha z}_{\mu_{s_\alpha z}}}$. Hence it follows from (2.3.2) that

$$(s_\alpha z)^{-1} \cdot (s_\alpha \cdot \mu_z) \geq (s_\alpha z)^{-1} \cdot \mu_{s_\alpha z}, \quad \text{i.e.,} \quad z^{-1} \cdot \mu_z \geq (s_\alpha z)^{-1} \cdot \mu_{s_\alpha z} = x^{-1} \cdot \mu_x.$$

Namely, we have shown that $z^{-1} \cdot \mu_z - x^{-1} \cdot \mu_x \in Q_+^\vee$, which, in particular, implies that for the fixed $j \in I$,

$$M_{z \cdot \Lambda_j} - M_{x \cdot \Lambda_j} = \langle \mu_z, z \cdot \Lambda_j \rangle - \langle \mu_x, x \cdot \Lambda_j \rangle = \langle z^{-1} \cdot \mu_z - x^{-1} \cdot \mu_x, \Lambda_j \rangle \geq 0,$$

as desired.

Case 2. Next we assume that $\langle \mu_z, \alpha \rangle \geq 0$. Recall that we have $(s_\alpha z)^{-1} \cdot \mu_z \geq (s_\alpha z)^{-1} \cdot \mu_{s_\alpha z}$ by the definition of GGMS data (see (2.2.1)). Also, we have $s_\alpha \cdot \mu_z = \mu_z - \langle \mu_z, \alpha \rangle \alpha^\vee$, where $\langle \mu_z, \alpha \rangle \geq 0$ by our assumption. Therefore, for the fixed $j \in I$ such that $\langle \alpha^\vee, z \cdot \Lambda_j \rangle \geq 0$, we compute:

$$\begin{aligned} 0 &\leq \big\langle (s_\alpha z)^{-1} \cdot \mu_z - (s_\alpha z)^{-1} \cdot \mu_{s_\alpha z}, \Lambda_j \big\rangle = \langle s_\alpha \cdot \mu_z, z \cdot \Lambda_j \rangle - \langle \mu_{s_\alpha z}, (s_\alpha z) \cdot \Lambda_j \rangle \\ &= \big\langle \mu_z - \langle \mu_z, \alpha \rangle \alpha^\vee, z \cdot \Lambda_j \big\rangle - M_{(s_\alpha z) \cdot \Lambda_j} = M_{z \cdot \Lambda_j} - \langle \mu_z, \alpha \rangle \langle \alpha^\vee, z \cdot \Lambda_j \rangle - M_{(s_\alpha z) \cdot \Lambda_j}, \end{aligned}$$

and hence obtain

$$M_{x\cdot\Lambda_j} = M_{(s_\alpha z)\cdot\Lambda_j} \leq M_{z\cdot\Lambda_j} - \underbrace{\langle \mu_z, \alpha \rangle}_{\geq 0} \underbrace{\langle \alpha^\vee, z \cdot \Lambda_j \rangle}_{\geq 0} \leq M_{z\cdot\Lambda_j},$$

as desired. This proves the proposition. □

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